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The notion of (p, r)-compactness

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Notations

p^*	The conjugate index of p (i.e.,$\frac{1}{p}+\frac{1}{p^*}=1$).
$\mathbb{K}$	The field of real or complex numbers.
$\mathcal{L}(X,Y)$	The space of all bounded linear operators from X to Y.
$\mathcal{K}(X,Y)$	The space of all compact operator from X to Y.
$\mathcal{K}_p(X,Y)$	The space of all p-compact operator from X to Y.
$\mathcal{F}(X,Y)$	The space of all finite-rank operators from X to Y.
$\overline{\mathcal{F}}(X,Y)$	The space of approximable operators from X to Y.
S_X	The unit sphere of X.
B_X	The unit ball of.
X^*	The topological dual of X.
$\ell_1(B_X)$	The Banach spaces of all absolutely summable scalar families (λ_x) where $x \in B_X$.
$\ell_\infty(B_{X^*})$	The Banach spaces of all bounded scalar families (λ_{x^*}) where $x^* \in B_{X^*}$.
Q_X	The natural surjection $Q_X : \ell_1(B_X) \to X$ is defined as $Q_X(\lambda_x)_{x\in B_X} = \sum\limits_{x\in B_X} \lambda_x x$.
J_X	The natural embedding $J_X : X \to \ell_\infty(B_{X^*})$ is defined as $J_X(x) = (x^*(x))_{x^*\in B_{X^*}}$.
$\mathrm{conv}\{(x_n)_n\}$	The convex hull of a sequence $(x_n)_n$.
$\overline{\mathrm{absconv}}\{(x_n)_n\}$	the closed absolutely convex hull of $(x_n)_n$.
p-$\mathrm{conv}\{(x_n)_n\}$	The p-convex hull of a sequence $(x_n)_n$.
$B_\varepsilon(x_0)$	The open ball centered at $x_0$with radius ε.
$\mathrm{Lip}_0(X,E)$	The space of all Lipschitz mappings T from X to E that vanish at 0.
$X^\# = \mathrm{Lip}_0(X,\mathbb{K})$	The Lipschitz dual of the pointed metric spaces X.
$\mathrm{Lip}_0(X,E)$	The space of Lipschitz compact operators.
$\mathcal{F}(X)$	The Lipschitz-free space of a metric space X.
Æ(X)	The Arens-Ells space.
T_L	The linearization of the Lipschitz operator.
$T^\#$	The Lipschitz adjoint (or dual) of T.
T^*	The adjoint linear operator of T.
T^t	The Lipschitz transpose map of T.

Introduction

In 1955, Grothendieck [12] described relatively compact subsets of a Banach space as sets contained in the closed convex hull of a norm null sequence. Nowadays this result is called the Grothendieck compactness principle. If one replaces null sequence with the p-summable sequences in the Grothendieck compactness principle, for some fixed real number $p \geq 1$, then one obtains a stronger form of relative compactness. This form of compactness was occasionally considered in 1980 by Reinov [19] and Bourgain and Reinov [6]. In this case, we say that K is relatively p-compact in the sense of Bourgain Reinov. In 2002 Sinha and Karn defined and studied in [21] another form of relative compactness, which lays between the aforementioned types of relative compactness. They required the set to belong to the so-called p-convex hull of a p-summable sequence. This type of sets are called in the present work relatively p-compact sets in the sense of Sinha Karn. Recently, Achour, Dahia, and Turco they extended the notion of p-compact operators in the sense of Sinha Karn which has been studied in [21] to Lipschitz p-compact operators [1].

In [4], Ain, Lillemets, and Oja they defined (p, r)-compact operators in an obvious way: a linear operator $T : X \to Y$ is (p, r)-compact if $T(B_X)$ is a relatively (p, r)-compact subset of Y. The purpose of this work is to study the class of (p, r)-compact operators, and describe its structure as an operator ideal. We also introduce the Lipschitz (p, r)-compact operators. Certain results and properties of this new class will be obtained.

This memoire has been organized as follows. In the first Chapter is an overview of notions and basic concepts and results needed in the following chapters. These include vector-valued sequence spaces, also operator ideals and we describe the ideal of (t, u, v)-nuclear operators. In the second Chapter We start by the notion of relatively (p, r)-compact sets, and we define the operator ideal $\mathcal{K}_{(p,r)}$ of all (p, r)-compact operators as an operator ideal. We prove that

$\mathcal{K}_{(p,r)}$ is a surjective operator ideal (see proposition 2.2.2). Also, we prove that $\mathcal{K}_{(p,r)}$ equal to $\mathcal{N}^{sur}_{(p,1,r^*)}$, the surjective hull of $(p,1,r^*)$-nuclear operators (see Theorem 2.2.1). We equip $\mathcal{K}_{(p,r)}$ with the corresponding s-norm from $\mathcal{N}^{sur}_{(p,1,r^*)}$ and we present the relationship between $\mathcal{K}_{(p,r)}$ and some operator ideals.

In the last Chapter we study the Lipschitz operators between pointed metric spaces and Banach spaces which are determined by (p,r)-compact sets. For this, we introduce three different notions of Lipschitz operators related with (p,r)-compact sets: the Lipschitz (p,r)-compact operators, the Lipschitz free-(p,r)-compact operators, and the locally (p,r)-compact Lipschitz operators.

Chapter 1

Preliminaries

In this chapter we recall some basic information and concepts used in the following chapter. Need the space of p-summable sequences, based on the book [9] by Diestel, Jarchow, and Tonge, also operator ideals with some ways to construct new operator ideals. We refer to Pietsch's monographs [17], [18] for the theory of operator ideals. Some preliminaries are also due to [4].

1.1 p-summable sequences

In this work the space of absolutely and weakly p-summable sequences are necessary. We introduce them in this section. Let X be a Banach space over $\mathbb{K}$, and $1 \leq p \leq \infty$. We start by The classical Banach sequence space ℓ_p, ℓ_∞, and c_0 are defined by

$$\begin{aligned}
\ell_p &= \ell_p(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty \right\} \\
\ell_\infty &= \ell_\infty(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_\infty = \sup_n |x_n| < \infty \right\} \\
c_0 &= c_0(\mathbb{K}) = \left\{ (x_n)_n \subset \mathbb{K} : \lim_{n\to\infty} |x_n| = 0 \right\}
\end{aligned}$$

Definition 1.1.1 *Let $1 \leq p < \infty$. The space $\ell_p(X)$ of absolutely p-summable sequences in X becomes a Banach space when equipped with the norm given by*

$$\|(x_n)_n\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}}.$$

Definition 1.1.2 *The space $\ell_\infty(X)$ of bounded sequences in X, and the space $c_0(X)$ of norm null sequences in X, are Banach spaces with the norm given by*

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

Definition 1.1.3 *Let $1 \leq p < \infty$. A sequence $(x_n)_n$ in X is said to be weakly p-summable if*

$$\sum_{n=1}^{\infty} |x^*(x_n)| < \infty,$$

for every $x^* \in B_{X^*}$.

We denote by $\ell_p^w(X)$ the Banach space of weakly p-summable sequences in X becomes a Banach space when equipped with the norm given by

$$\|(x_n)_n\|_p^w = \sup \left\{ (\sum_{n=1}^{\infty} |x^*(x_n)|^p)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

Remark 1.1.1 *In the case* $p = \infty$. *Then the space* $\ell_\infty^w(X)$ *of weakly bounded sequences coincide with the space* $\ell_\infty(X)$ *and*

$$\|(x_n)_n\|_\infty^w = \|(x_n)_n\|_\infty .$$

Definition 1.1.4 *A sequence* $(x_n)_n$ *in* X *is said to be weakly null if*

$$\lim_{n\to\infty} |x^*(x_n)| = 0 \ ,$$

for every $x^* \in X^*$.

We denote by $c_0^w(X)$ *the Banach space of weakly null sequences in* X *is a closed subspace of* $\ell_\infty(X)$, *therefore, it is a Banach space with the supremum norm of* $\ell_\infty(X)$.

1.2 Operator ideals

In this section, we recall some basic facts and properties about operator ideals. We also recall some of the classical examples.

Let X and Y be Banach spaces. It is well known that an operator $T \in \mathcal{L}(X, Y)$ is of finite rank if and only if there exist functionals $x_1^*, ...x_n^* \in X^*$, and vectors $y_1, ...y_n \in Y$ such that

$$T(x) = \sum_{k=1}^{n} x_k^*(x) y_k, x \in X.$$

The class of all Finite rank operator is denoted by $\mathcal{F}$.

Following the standard tensor-product notation, for $x^* \in X^*$ and $y \in Y$ the operator $x \longmapsto x^*(x)y$, $x \in X$, is denoted by $x^* \otimes y$. It is clear that $x^* \otimes y$ is a rank one operator if and only if $x^* \neq 0$ and $y \neq 0$.

Therefore, $T \in \mathcal{F}(X, Y)$ if and only if T can be represented as a finite sum of rank one operators

$$T = \sum_{k=1}^{n} x_k^* \otimes y_k.$$

Definition 1.2.1 *An operator ideal* $\mathcal{A}$ *is a subclass of* $\mathcal{L}$ *such that the components*

$$\mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(X, Y),$$

satisfy the following conditions:

(i) $\mathcal{A}(X,Y)$ is a linear subspaces of $\mathcal{L}(X,Y)$ for all Banach spaces X and Y.

(ii) The subclass $\mathcal{A}$ contains all Finite rank operator.

(iii) The ideal property: if X, Y, Z, W are Banach spaces and $R \in \mathcal{L}(X,Y)$, $T \in \mathcal{A}(Y,Z)$, $S \in \mathcal{L}(Z,W)$, then $STR \in \mathcal{A}(X,W)$.

If $\|.\|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathbb{R}_+$ satisfies

(i') $(\mathcal{A}(X,Y), \|.\|_{\mathcal{A}})$ is a normed (Banach) spaces for all Banach spaces X and Y.

(ii') $\|Id_{\mathbb{K}}\|_{\mathcal{A}} = 1$.

(iii') If $R \in \mathcal{L}(X,Y)$, $T \in \mathcal{A}(Y,Z)$, $S \in \mathcal{L}(Z,W)$, then

$$\|STR\|_{\mathcal{A}} \leq \|S\| \, \|T\| \, \|R\| \, ,$$

then $(\mathcal{A}, \| \, . \, \|_{\mathcal{A}})$ is called normed (Banach) operators ideal.

Examples

1. **Approximable operators.** An operator $T \in \mathcal{L}(X,Y)$ is called Approximable operators if there are $T_n \in \mathcal{F}(X,Y)$, with

$$\lim_{n} \|T - T_n\| = 0.$$

We denote by $\overline{\mathcal{F}}(X,Y)$ the ideal space of all approximable operators from X to Y.

2. **Compact linear operators.** A linear operator $T \in \mathcal{L}(X,Y)$ is compact if the image of the unit ball B_X of X into a relatively compact subset of Y.

In other words, T is compact if and only if for every norm bounded sequence $\{x_n\}$ of X, the sequence $\{T(x_n)\}$ has a norm convergent subsequence in Y.

We denote by $\mathcal{K}(X,Y)$ the ideal space of compact operators from X to Y.

Definition 1.2.2 *A map* $\|\ .\ \|$: $X \to \mathbb{R}_+$ *is a quasi-norm if the following conditions are satisfied:*

(1) $\|x\| = 0 \Longleftrightarrow x = 0$.

(2) There exists $k \geq 1$ such that $\|x + y\| \leq k(\|x\| + \|y\|)$ for $x,\ y \in X$.

(3) $\|\lambda x\| = |\lambda|\, \|x\|$ for $x \in X,\ \lambda \in \mathbb{K}$.

Definition 1.2.3 *Let* $0 < s \leq 1$. *A quasi-norm is called an s-norm, if condition* (2) *is replaced with the s-triangle inequality*

$$\|x + y\|^s \leq \|x\|^s + \|y\|^s \quad x,\ y \in X$$

Remark 1.2.1 *A* 1*-norm is just a norm, and an s-norm is also a t-norm if* $0 < t < s \leq 1$.

Every quasi-norm is equivalent to an s-norm, where $k = 2^{\frac{1}{s}-1}$. An s-norm induces a metric topology on X that can be defined by

$$d(x, y) = \|x - y\|^s .$$

The space X is said to be s-Banach spaces if it is complete for this metric (see [14]).

Definition 1.2.4 *Let* $0 < s \leq 1$. *An operator ideal* $\mathcal{A}$ *is called an s-Banach operator ideal and denoted by* $(\mathcal{A},\ \|\ .\ \|_{\mathcal{A}})$ *if there exists a function* $\|\ .\ \|_{\mathcal{A}}$: $\mathcal{A} \to \mathbb{R}_+$ *called an s-norm, such that:*

1. *All components* $\mathcal{A}(X, Y) = (\mathcal{A}(X, Y), \|\ .\ \|_{\mathcal{A}})$ *are s-Banach spaces.*

2. $\|x^* \otimes y\|_{\mathcal{A}} = \|x^*\|\, \|y\|$ *for all rank one operators* $x^* \otimes y$.

3. *If X,Y,Z,W are Banach spaces and* $R \in \mathcal{L}(X, Y)$, $T \in \mathcal{A}(Y, Z)$, $S \in \mathcal{L}(Z, W)$, *then*

$$\|STR\|_{\mathcal{A}} \leq \|S\|\, \|T\|_{\mathcal{A}}\, \|R\| .$$

For an operator ideal $\mathcal{A}$ there are several ways to produce new operator ideals.
(see [18])

- The components of $\mathcal{A}^{sur}$, the surjective hull of $\mathcal{A}$ are defined by

$$\mathcal{A}^{sur}(X,Y) = \{T \in \mathcal{L}(X,Y) : TQ_X \in \mathcal{A}(\ell_1(B_X),Y)\}.$$

Where The natural surjection $Q_X : \ell_1(B_X) \to X$ is defined as

$$Q_X((\lambda_x)_{x\in B_X}) = \sum_{x\in B_X} \lambda_x x,\ (\lambda_x)_{x\in B_X} \in \ell_1(B_X).$$

If $(\mathcal{A}, \| \,.\, \|_{\mathcal{A}})$ are s-Banach spaces ideal, then $\mathcal{A}^{sur}$ is also an s-Banach operator ideal with

$$\|T\|_{\mathcal{A}^{sur}} = \| TQ_X \|_{\mathcal{A}},$$

for $T \in \mathcal{A}^{sur}(X,Y)$.

- The components of $\mathcal{A}^{inj}$, the injective hull of $\mathcal{A}$ are defined by

$$\mathcal{A}^{inj}(X,Y) = \{T \in \mathcal{L}(X,Y) : J_Y T \in \mathcal{A}(X, \ell_\infty(B_{Y^*})\}.$$

Where The natural embedding $J_Y : Y \to \ell_\infty(B_{Y^*})$ is defined as

$$J_Y(y) = (y^*(y))_{y^*\in B_{Y^*}}, y \in Y.$$

If $(\mathcal{A}, \| \,.\, \|_{\mathcal{A}})$ are s-Banach spaces ideal, then $\mathcal{A}^{inj}$ is also an s-Banach operator ideal with

$$\|T\|_{\mathcal{A}^{inj}} = \| J_Y T \|_{\mathcal{A}},$$

for $T \in \mathcal{A}^{inj}(X,Y)$.

- The components of $\mathcal{A}^{dul}$, the dual of $\mathcal{A}$ are defined by

$$\mathcal{A}^{dul}(X,Y) = \{T \in \mathcal{L}(X,Y) : T^* \in \mathcal{A}(Y^*, X^*)\}.$$

If $(\mathcal{A}, \| \,.\, \|_{\mathcal{A}})$ are s-Banach spaces ideal, then $\mathcal{A}^{dul}$ is also an s-Banach operator ideal with

$$\|T\|_{\mathcal{A}^{dul}} = \|T^*\|_{\mathcal{A}},$$

for $T \in \mathcal{A}^{dul}(X,Y)$.

- The components of $\mathcal{A}^{reg}$, the regular hull of $\mathcal{A}$ are defined by

$$\mathcal{A}^{reg}(X,Y) = \{T \in \mathcal{L}(X,Y) : J_Y T \in \mathcal{A}(X, Y^{**})\}.$$

If $(\mathcal{A}, \| \, . \, \|_{\mathcal{A}})$ are s-Banach spaces ideal, then $\mathcal{A}^{reg}$ is also an s-Banach operator ideal with

$$\|T\|_{\mathcal{A}^{reg}} = \|J_Y T\|_{\mathcal{A}},$$

for $T \in \mathcal{A}^{reg}(X,Y)$.

Clearly $\mathcal{A} \subset \mathcal{A}^{sur}$, $\mathcal{A} \subset \mathcal{A}^{inj}$.

If $\mathcal{A} = \mathcal{A}^{sur}$, (respectively, $\mathcal{A} = \mathcal{A}^{inj}$) as ($s$-Banach) operator ideals, then $\mathcal{A}$ is said to be a surjective (respectively, an injective) (s-Banach) operator ideal.

1.3 The ideal of (t, u, v)-nuclear operators

Let X, and Y be Banach space.

Definition 1.3.1 *[17, 18.1.1] Let $0 < t \leq \infty$, $1 \leq u$, $v \leq \infty$, and $\frac{1}{u} + \frac{1}{v} \leq 1 + \frac{1}{t}$. An operator $T \in \mathcal{L}(X,Y)$ is called (t,u,v)-nuclear if*

$$T = \sum_{n=1}^{+\infty} \sigma_n x_n^* \otimes y_n \tag{1.3.1}$$

with $(\sigma_n)_n \in \ell_t$, $(x_n^*)_n \in \ell_{v^*}^w(X^*)$, and $(y_n)_n \in \ell_{u^*}^w(Y)$. In the case $t = \infty$, let us suppose that $(\sigma_n)_n \in c_0$.

The class of all (t,u,v)-nuclear operators is denoted by $\mathcal{N}_{(t,u,v)}$.

We put

$$N_{(t,u,v)}(T) = \|T\|_{\mathcal{N}(t,u,v)} := \inf \|(\sigma_n)_n\|_t \, \|(x_n^*)_n\|_{v^*}^w \, \|(y_n)_n\|_{u^*}^w$$

where the infimum is taken over all (t,u,v)-nuclear representations (3.2.4).

Theorem 1.3.1 *[17, 18.1.2] Let*

$$\frac{1}{s} = \frac{1}{u^*} + \frac{1}{v^*} + \frac{1}{t},$$

then $(\mathcal{N}_{(t,u,v)}, \| \, . \, \|_{\mathcal{N}_{(t,u,v)}})$ is an s-Banach operator ideal.

we now characterize (t, u, v)-nuclear operators by a factorization property.

Theorem 1.3.2 *[17, 18.1.3] An operator $T \in \mathcal{L}(X, Y)$ is called (t, u, v)-nuclear if and only if there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell_{v^*} & \underset{\Delta}{\longrightarrow} & \ell_u \end{array}$$

such that $\Delta \in \mathcal{L}(\ell_{v^*}, \ell_u)$ is a diagonal operator of the form $\Delta(a_n)_n = (a_n\sigma_n)_n$,with $(\sigma_n)_n \in \ell_t((\sigma_n)_n \in c_0$ when $t = \infty)$, $A \in \mathcal{L}(X,\, \ell_{v^*})$, $B \in \mathcal{L}(\ell_u, Y)$, and

$$\|T\|_{\mathcal{N}_{(t,u,v)}} = \inf \;\; \|B\| \, \|(\sigma_n)_n\| \, \|A\| \,,$$

where the infimum is taken over the all possible factorizations.

Let $1 \le p \le \infty$, $1 \le r \le p^*$. For every $x = (x_n)_n \in \ell_p(X)$ defines an operator $\Phi_x \in \mathcal{L}(\ell_r, X)$ (see [4]) through the equality

$$\Phi_x(\alpha) = \sum_{n=1}^{+\infty} \alpha_n x_n \;, \quad \alpha = (\alpha_n)_n \in \ell_r.$$

Let $(e_n)_n$ be the unit vector basis of $\ell_{r^*} \subset (\ell_r)^*$ (c_0 when $r = 1$), considered as coordinate functionals for ℓ_r. Then we clearly have

$$\Phi_x = \sum_{n=1}^{+\infty} e_n \otimes x_n.$$

Proposition 1.3.1 *Let $1 \le p \le \infty$, $1 \le r \le p^*$. Let $x = (x_n)_n \in \ell_p(X)$ $((x_n)_n \in c_0(X)$ when $p = \infty)$. Then the operator $\Phi_x : \ell_r \to X$ is approximable, i.e., $\Phi_x \in \overline{\mathcal{F}}(\ell_r, X)$.*

Proof. Let $(x_n)_{n \le m} := (x_1, ..., x_m, ...0, 0, ...)$, we have

$$\Phi_x = \sum_{n=1}^{+\infty} e_n \otimes x_n.$$

Now, since $1 \le r \le p^*$,

$$
\begin{aligned}
\left\|\Phi_x - \Phi_{(x_n)_{n\le m}}\right\| &= \sup_{(\alpha_n)_n\in B_{\ell_r}} \left\|\sum_{n=m+1}^{\infty} \alpha_n x_n\right\| \\
&\le \sup_{(\alpha_n)_n\in B_{\ell_r}} \sum_{n=m+1}^{\infty} |\alpha_n|\,\|x_n\| \\
&\quad \sup_{(\alpha_n)_n\in B_{\ell_r}} \sup_{n\ge m+1} |\alpha_n| \sum_{n=m+1}^{\infty} \|x_n\| \qquad \text{if } \ p=1 \\
&\le \sup_{(\alpha_n)_n\in B_{\ell_r}} \Big(\sum_{n=m+1}^{\infty} |\ \alpha_n\ |^{p^*}\Big)^{\frac{1}{p^*}} \Big(\sum_{n=m+1}^{\infty} \|x_n\|^{p}\Big)^{\frac{1}{p}} \quad \text{if } \ 1<p<\infty \\
&\quad \sup_{(\alpha_n)_n\in B_{\ell_1}} \sup_{n\ge m+1} \|x_n\| \sum_{n=m+1}^{\infty} |\alpha_n| \qquad \text{if } \ p=\infty \\
&\le \left\{ \begin{array}{ll} \left(\sum\limits_{n=m+1}^{\infty} \|x_n\|^p\right)^{\frac{1}{p}} & \text{if } \ 1\le p<\infty \\ \sup\limits_{n\ge m+1} \|x_n\| & \text{if } \quad p=\infty \end{array} \right. \underset{m}{\longrightarrow} 0
\end{aligned}
$$

Thus Φ_x converges to $\Phi_{(x_n)_{n\le m}}$ in $\mathcal{L}(\ell_r, X)$ as $m\to\infty$. Then $\Phi_x \in \overline{\mathcal{F}}(\ell_r, X)$. ■

Remark 1.3.1 *Since $\overline{\mathcal{F}} \subset \mathcal{K}$, we immediately get that $\Phi_x \in \mathcal{K}(\ell_r, X)$. for $x=(x_n)_n \in \ell_p(X)$ ($(x_n)_n \in c_0(X)$ when $p=\infty$).*

Proposition 1.3.2 *Let $1\le p\le\infty$, $1\le r\le p^*$. Let $x=(x_n)_n \in \ell_p(X)$ ($(x_n)_n\in c_0(X)$ when $p=\infty$). Then the operator $\Phi_x : \ell_r \to X$ is $(p,1,r^*)$-nuclear, i.e.,*

$$\Phi_x \in \mathcal{N}_{(p,1,r^*)},$$

and

$$\|\Phi_x\|_{\mathcal{N}_{(p,1,r^*)}} \le \|x\|_p\,.$$

Proof. For $\Phi_x \in \mathcal{K}(\ell_r, X)$ we have

$$\Phi_x = \sum_{n=1}^{+\infty} e_n \otimes x_n.$$

Where $(e_n)_n$ be the unit vector basis of $\ell_{r^*} \subset (\ell_r)^*$ are considered as coordinate functionals for ℓ_r. It is well known and easy to verify that $(e_n)_n \in S_{\ell_r^w(\ell_{r^*})}$. Therefore, from $\|x_n\|\, e_n \otimes (\|x_n\|^{-1} x_n) = e_n \otimes x_n$ it is clear that $\Phi_x \in \mathcal{N}_{(p,1,r^*)}$ and

$$\|\Phi_x\|_{\mathcal{N}_{(p,1,r^*)}} \le \|x\|_p\,,$$

as desired. ■

We recall that the opertator $\overline{\Phi}_x$ is the injective associate of Φ_x defined by $\overline{\Phi}_x : Z := \ell_r / \ker \Phi_x \to X$ satisfy

$$\Phi_x = \overline{\Phi}_x q,$$

where $q : \ell_r \to Z := \ell_r / \ker \Phi_x$ is the quotient mapping. Then we have the following

Proposition 1.3.3 *[4, p.149] Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. Let $x = (x_n)_n \in \ell_p(X)$ ($(x_n)_n \in c_0(X)$ when $p = \infty$). Then the operator*

$$\overline{\Phi}_x \in \mathcal{N}^{sur}_{(p,1,r^*)}(Z, X),$$

and

$$\left\|\overline{\Phi}_x\right\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \leq \|x\|_p .$$

Proof. Let $\epsilon > 0$ be arbitrary. For every $z \in B_Z$ choose $\alpha_z \in B_{\ell_r}$ with $\|\alpha_z\| \leq 1+\epsilon$ such that $q(\alpha_z) = z$. Define $\widehat{Q} : \ell_1(B_Z) \to \ell_r$ by $\widehat{Q}((\lambda_z)_{z \in B_Z}) = \sum\limits_{z \in B_Z} \lambda_z \alpha_z$. Then $\widehat{Q} \in \mathcal{L}(\ell_1(B_Z), \ell_r)$, $\left\|\widehat{Q}\right\| \leq 1 + \epsilon$ and $q\widehat{Q} = Q_z$:

$$\begin{tikzcd}
 & \ell_r \arrow[rr, "\Phi_x"] \arrow[dr, "q"] & & X \\
\ell_1(B_Z) \arrow[ur, "\hat{Q}"] \arrow[rr, "Q_Z"] & & Z \arrow[ur, "\overline{\Phi}_x"'] &
\end{tikzcd}$$

Therefore,

$$\overline{\Phi}_x Q_z = \Phi_x \widehat{Q} \in \mathcal{N}_{(p,1,r^*)}(\ell_1(B_Z), X),$$

meaning that $\overline{\Phi}_x \in \mathcal{N}^{sur}_{(p,1,r^*)}(Z, X)$. Moreover,

$$\begin{aligned}
\left\|\overline{\Phi}_x\right\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} &= \left\|\Phi_x \widehat{Q}\right\|_{\mathcal{N}_{(p,1,r^*)}} \\
&\leq \|\Phi_x\|_{\mathcal{N}_{(p,1,r^*)}} \left\|\widehat{Q}\right\| \\
&\leq (1+\epsilon) \|\Phi_x\|_{\mathcal{N}_{(p,1,r^*)}} \\
&\leq (1+\epsilon) \|(x_n)_n\| .
\end{aligned}$$

Since this holds for every $\epsilon > 0$, we have $\left\|\overline{\Phi}_x\right\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \leq \|(x_n)_n\|$. ∎

Chapter 2

The notion of (p, r)-compactness

In this chapter, we define the class of (p, r)-compact operators and describe its structure as an operator ideal. This chapter is mainly based on [4].

2.1 Relatively (p,r)-compact subsets

Let X be a Banach space.

Definition 2.1.1 *The convex hull of a sequence $(x_n)_n \in c_0(X)$ is defined as*

$$conv\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\}$$

Definition 2.1.2 *Let $1 \leq p \leq \infty$. The p-convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as*

$$p - \text{conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p^*}} \right\}.$$

Definition 2.1.3 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. The (p,r)-convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as*

$$(p,r) - conv\{(x_n)_n\} = \left\{ \sum_{n=1}^{+\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_r} \right\}.$$

If $(x_n)_n \in \ell_\infty(X)$, then $(\infty,1)-conv\{(x_n)_n\}$is exactly $\overline{absconv}\{(x_n)_n\}$, the closed absolutely convex hull of $(x_n)_n$, when

$$\overline{absconv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{m} \alpha_n x_n : \sum_{n=1}^{m} \mid \alpha_n \mid \leq 1,\ m \in \mathbb{N} \right\}.$$

Remark 2.1.1 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. We have*

$$(p,r) - conv\{(x_n)_n\} = \Phi_x(B_{\ell_r}). \tag{2.1.1}$$

Theorem 2.1.1 *[12]**(Grothendieck compactness principle)**. A subset K of X is relatively compact if and only if there exists $(x_n)_n \in c_0(X)$ such that $K \subset \overline{conv}\{(x_n)_n\}$, the closed convex hull of the sequence $(x_n)_n$.*

*If we consider X to be a finite-dimensional space, this theorem is equivalent to the **Heine-Borel**, as in such spaces a closed subset of the closed convex hull of a norm null sequence is precisely a closed and bounded subset.*

Let $p \geq 1$. If on replaces the space $c_0(X)$ with $\ell_p(X)$ in the Grothendieck compactness principle, then one obtains a stronger form of relative compactness is called relatively p-compact in the sense of Bourgain Reinov see [19], [6].

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. In [21] another strong form of compactness was introduced by Sinha and Karn through the requirement that $K \subset p - conv\{(x_n)_n\}$ for some $(x_n)_n \in \ell_p(X)$. In this case, we say that K is relatively p-compact in the sense of Sinha Karn.

In [4] they introduced the notion of relatively (p, r)-compact sets in the following way.

Definition 2.1.4 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A subset K of X is called relatively (p, r)-compact if*

$$K \subset (p, r) - conv\{(x_n)_n\},$$

for some $(x_n)_n \in \ell_p(X)((x_n)_n \in c_0(X)$ when $p = \infty)$.

Remark 2.1.2 *We have*

1. The $(p, 1)$-compactness is precisely the Bourgain Reinov p-compactness.
2. The (p, p^*)-compactness is precisely the Sinha Karn p-compactness.
3. According to Grothendieck compactness principle, the $(\infty, 1)$-compactness coincides with the compactness because $(\infty, 1) - conv\{(x_n)_n\}$ is precisely the closed absolutely convex hull of $(x_n)_n$.

Remark 2.1.3 *Definition 2.1.4 means by (2.1.1) that a subset K of X is relatively (p, r)-compact if and only if*

$$K \subset \Phi_x(B_{\ell_r}),$$

for some $x = (x_n)_n \in \ell_p(X)$.

In particular, $\Phi_x(B_{\ell_r})$ itself is relatively (p, r)-compact.

Theorem 2.1.2 *Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. Assume that*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r}. \tag{2.1.2}$$

If a subset of X is relatively (p, r)-compact, then it is relatively (q, s)-compact.

Proof. If $q = \infty$. Then $s = 1$. We know that the $(\infty, 1)$-compactness coincides with the usual compactness. Thus, we have to show that relatively (p, r)-compact sets are relatively compact. For this it is sufficient to show that $\Phi_x(B_{\ell_r})$ is relatively compact. But the latter immediately follows from the compactness of Φ_x.

If $1 \le p \le q < \infty$, and $r \le s$, then the assertion in clear as $\ell_p(X) \subset \ell_q(X)$ and $B_{\ell_r} \subset B_{\ell s}$, hence

$$\Phi_x(B_{\ell_r}) \subset \Phi_x(B_{\ell_s})$$

for any $x = (x_n)_n \in \ell_p(X)$.

Finally, let $1 \le p \le q < \infty$, and $s < r$, then if $(x_n)_n \in \ell_p(X)$, then

$$y = (y_n)_n := (c \left\| x_n \right\|^{\frac{p-q}{q}} x_n)_n \in \ell_q(X)$$

where $c := \left\| x_n \right\|^{\frac{q-p}{q}}$.

For $a = (a_n)_n \in B_{\ell r}$, put

$$b_n := c^{-1} \left\| x_n \right\|^{\frac{q-p}{q}} a_n$$

so that $b_n y_n = a_n x_n$, $n \in \mathbb{N}$. We shall show that $b = (b_n)_n \in B_{\ell s}$.

Assume first that $r < \infty$. Then $\frac{r}{s} > 1$ and $(\frac{r}{s})^* = \frac{r}{r-s}$ and we can apply Hölder's inequality to obtain

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} | \, b_n \, |^s \right)^{\frac{1}{s}} &= \left(\sum_{n=1}^{\infty} | \, c^{-1} \left\| x_n \right\|^{\frac{q-p}{q}} a_n \, |^s \right)^{\frac{1}{s}} \\
&\le \left(\left(\sum_{n=1}^{\infty} | \, a_n \, |^r \right)^{\frac{s}{r}} \left(\sum_{n=1}^{\infty} | \, c^{\frac{sr}{s-r}} \left\| x_n \right\|^{\frac{q-p}{q} \frac{sr}{r-s}} \, | \right)^{\frac{r-s}{r}} \right)^{\frac{1}{s}} \\
&\le \left(\sum_{n=1}^{\infty} | \, a_n \, |^r \right)^{\frac{1}{r}} \left(\sum_{n=1}^{\infty} | \, c^{\frac{sr}{s-r}} \left\| x_n \right\|^{\frac{q-p}{q} \frac{sr}{r-s}} \, | \right)^{\frac{r-s}{sr}} \\
&\le c^{-1} \left(\left\| (x_n)_n \right\|_{\frac{q-p}{q} \frac{sr}{r-s}} \right)^{\frac{q-p}{q}} \\
&\le c^{-1} \left\| (x_n)_n \right\|_p^{\frac{q-p}{q}} = 1
\end{aligned}$$

because, due to (2.1.2),

$$p \le \frac{q-p}{q} \frac{sr}{r-s} < \infty.$$

If $r=\infty$, then $p=1$. Hence by (2.1.2), $s=q^*$ and

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \mid b_n \mid^s\right)^{\frac{1}{s}} &= \left(\sum_{n=1}^{\infty} \mid c^{-1}\, \|x_n\|^{\frac{q-1}{q}}\, a_n \mid^{q^*}\right)^{\frac{1}{q^*}} \\ &\leq c^{-1}\left(\sum_{n=1}^{\infty} \|x_n\|^{\frac{q-1}{q}q^*}\right)^{\frac{1}{q^*}} \\ &\leq c^{-1}\left(\sum_{n=1}^{\infty} \|x_n\|\right)^{\frac{q-1}{q}} = c^{-1}c = 1.\end{aligned}$$

Since

$$\Phi_x(\alpha) = \sum_{n=1}^{+\infty}\alpha_n x_n = \sum_{n=1}^{+\infty} b_n y_n,$$

and $(b_n)_n \in B_{\ell s}$, we have

$$\Phi_x(B_{\ell_r}) \subset \Phi_y(B_{\ell_s}).$$

Showing that the (p,r)-compactness implies the (q,s)-compactness also in this case. ■

2.2 The operators ideal of (p,r)-compact operators

Let X, Y be Banach spaces. Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$.

The main objective of this section is to define the class $\mathcal{K}_{(p,r)}$ of (p,r)-compact operator and describe its structure as an operator ideal. Following [21], denote the class of all p-compact operators in the sense of Sinha-Karn by $\mathcal{K}_p$. Properties of $\mathcal{K}_p$ were studied in [21, 15], and the references therein.

Definition 2.2.1 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A linear operator $T : X \to Y$ is (p,r)-compact if $T(B_X)$ is a relatively (p,r)-compact subset of Y.*

Denote the class of all (p,r)-compact operators acting between arbitrary Banach spaces by $\mathcal{K}_{(p,r)}$.

Remark 2.2.1 *It is clear that $\mathcal{K}_{(\infty,1)} = \mathcal{K}$, $\mathcal{K}_{(p,p^*)} = \mathcal{K}_p$, and the class of p-compact operators in the sense of Bourgain Reinov is precisely $\mathcal{K}_{(p,1)}$.*

Remark 2.2.2 *Let $T : X \to Y$ be a linear operator. From (2.1.1) it is clear that $T \in \mathcal{K}_{(p,r)}(X,Y)$ if and only if $T(B_X) \subset \Phi_y(B_{\ell_r})$ for some $y = (y_n)_n \in \ell_p(Y)((y_n)_n \in c_0$ when $p=\infty)$.*

Proposition 2.2.1 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. The class of (p, r)-compact operators $\mathcal{K}_{(p,r)}$ is an operator ideal.*

Proof. $\mathcal{K}_{(p,r)}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$. Let $S, T \in \mathcal{K}_{(p,r)}(X, Y)$ be such that

$$\begin{cases} S(B_X) \subset \Phi_x(B_{\ell_r}) \\ T(B_X) \subset \Phi_y(B_{\ell_r}) \end{cases}$$

for some $x = (x_n)_n, y = (y_n)_n \in \ell_p(Y)((x_n)_n, (y_n)_n \in c_0(Y)$ when $p = \infty)$, and let $a \in \mathbb{K}$. Put

$$z_n = \begin{cases} 2^{\frac{1}{r}} a x_{\frac{(n+1)}{2}} & \text{if} \quad n \text{ is odd} \\ 2^{\frac{1}{r}} y_n & \text{if} \quad n \text{ is even} \end{cases}$$

It is easy to verify that then $z = (z_n)_n \in \ell_p(Y)((z_n)_n \in c_0(Y)$ when $p = \infty)$,

and $(aS + T)(B_X) \subset \Phi_z(B_{\ell_r})$, meaning that $(aS + T)(B_X) \in \mathcal{K}_{(p,r)}(X, Y)$.

This shows that $\mathcal{K}_{(p,r)}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$.

The space $\mathcal{K}_{(p,r)}(X, Y)$ contains all rank one operators. We have, $x^* \otimes y \in \mathcal{K}_{(p,r)}(X, Y)$ because

$$(x^* \otimes y)(B_X) \subset \Phi_z(B_{\ell_r})$$

for $(z_n)_n$ with $z_1 = \| x^* \| y$, $z_2 = z_3 = ... = 0$.

Let $R \in \mathcal{L}(Z, X)$, $T \in \mathcal{K}_{(p,r)}(X, Y)$, $S \in \mathcal{L}(Y, W)$, then

$$STR \in \mathcal{K}_{(p,r)}(Z, W),$$

because

$$(STR)(B_z) \subset \Phi_w(B_{\ell_r})$$

for $w = (w_n)_n = (\| R \| S(y_n))_n$. ■

Proposition 2.2.2 *The operator ideal $\mathcal{K}_{(p,r)}$ is surjective, i.e. $\mathcal{K}_{(p,r)} = \mathcal{K}^{sur}_{(p,r)}$.*

Proof. Let $T \in \mathcal{K}^{sur}_{(p,r)}(X, Y)$. Denote $Z = \ell_1(B_X)$. Then $TQ_X \in \mathcal{K}_{(p,r)}(Z, Y)$, ($Q_X$ the natural surjection $Q_X : \ell_1(B_X) \to X$ is defined by $Q_X(\lambda_x)_{x \in B_X} = \sum\limits_{x \in B_X} \lambda_x x, (\lambda_x)_{x \in B_X} \in \ell_1(B_X)$), and there exists $(y_n)_n \in \ell_p(Y)((y_n)_n \in c_0(Y)$ when $p = \infty)$, such that

$$(TQ_X)(B_Z) \subset \Phi_y(B_{\ell_r}).$$

But $B_X \subset (Q_X)(B_Z)$ because for $x_0 \in B_X$ we have $x_0 = Q_X(\lambda_x)$,where

$$\lambda_x = \begin{cases} 1 & \text{if} \quad x = x_0 \\ 0 & \quad\text{otherwise} \end{cases}$$

Hence $T(B_X) \subset \Phi_y(B_{\ell_r})$. ■

Corollary 2.2.1 *Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. Assume that*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r}$$

Then (p,r)-compact operators are also (q,s)-compact operators ,i.e.

$$(\mathcal{K}_{(p,r)}, \|\,.\,\|_{\mathcal{K}_{(p,r)}}) \subset (\mathcal{K}_{(q,s)}, \|\,.\,\|_{\mathcal{K}_{(q,s)}}).$$

In particular if $1 \leq p \leq q \leq \infty$, then

$$(\mathcal{K}_{(p,1)}, \|\,.\,\|_{\mathcal{K}_{(p,1)}}) \subset (\mathcal{K}_{(q,1)}, \|\,.\,\|_{\mathcal{K}_{(q,1)}}).$$

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. If $T \in \mathcal{K}_{(p,r)}(X,Y)$. We have the natural factorization of T as follows.

$$T = \overline{\Phi}_y T_y$$

with $T_y \in \mathcal{L}(X,Z)$, and $\overline{\Phi}_y : Z \to Y$ is the injective associate of Φ_y, where $Z = \ell_r / \ker \Phi_y$

Theorem 2.2.1 $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. *Then $\mathcal{K}_{(p,r)} = \mathcal{N}^{sur}_{(p,1,r^*)}$.*

Proof. *Let $T \in \mathcal{K}_{(p,r)}(X,Y)$. Since, by the natural factorization, $T = \overline{\Phi}_y T_y$ and $\overline{\Phi}_y \in \mathcal{N}^{sur}_{(p,1,r^*)}(Z,Y)$, where $Z = \ell_r / \ker \Phi_y$ (see Proposition 1.3.3), we have $T \in \mathcal{N}^{sur}_{(p,1,r^*)}(X,Y)$. On the other hand, to see that $\mathcal{N}^{sur}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$, it suffices to prove that $\mathcal{N}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$, because $\mathcal{K}_{(p,r)}$ is surjective (see Proposition 2.2.2), and $\mathcal{A}^{sur} \subset \mathcal{B}^{sur}$ whenever $\mathcal{A} \subset \mathcal{B}$. Consider $T \in \mathcal{N}_{(p,1,r^*)}(X,Y)$. Then*

$$T = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n$$

with $(\sigma_n)_n \in \ell_p$ ($(\sigma_n)_n \in c_0$ when $p = \infty$), $(x_n^*)_n \in \ell_r^w(X^*)$, and $(y_n)_n \in \ell_\infty(Y)$. We clearly may assume that $\|(x_n^*)_n\|_r^w = 1$. Indeed,

$$\sigma_n x_n^* \otimes y_n = \sigma_n \frac{x_n^*}{\|(x_n^*)_n\|_r^w} \otimes \|(x_n^*)_n\|_r^w \, y_n$$

and $(\|(x_n^*)_n\|_r^w\, y_n)_n \in \ell_\infty(Y)$. *Observe that* $(\sigma_n y_n)_n \in \ell_p(Y)((\sigma_n y_n)_n \in c_0(Y))$ *for* $p = \infty$). *Together with the assumption* $\|(x_n^*)_n\|_r^w = 1$, *we have*

$$T(x) = \sum_{n=1}^{+\infty} \sigma_n x_n^*(x) y_n \in \Phi_{(\sigma_n y_n)_n}(B_{\ell_r})$$

for every $x \in B_X$. *Thus we have shown that* $T \in \mathcal{K}_{(p,r)}(X, Y)$. *Since* $\mathcal{K}_{(p,r)} = \mathcal{N}^{sur}_{(p,1,r^*)}$. ■

The equality $\mathcal{K}_{(p,r)} = \mathcal{N}^{sur}_{(p,1,r^*)}$ of operator ideals from Theorem 2.2.1 allows us to equip $\mathcal{K}_{(p,r)}$ with an s-norm in the following natural.

Definition 2.2.2 *Let* $1 \leq p \leq \infty$ *and* $1 \leq r \leq p^*$. *Define*

$$\| \,.\, \|_{\mathcal{K}_{(p,r)}} := \| \,.\, \|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \,.$$

Theorem 2.2.2 *Let* $1 \leq p \leq \infty$ *and* $1 \leq r \leq p^*$. *Then* $(\mathcal{K}_{(p,r)}, \| \,.\, \|_{\mathcal{K}_{(p,r)}})$ *is an* s*-Banach operator ideal, and* $\mathcal{K}_{(p,r)} = \mathcal{N}^{sur}_{(p,1,r^*)}$ *as* s*-Banach operator ideals, where* s *satisfies*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{r}.$$

Theorem 2.2.3 *Let* $1 \leq p \leq \infty$, $1 \leq r \leq p^*$, *and* $T \in \mathcal{K}_{(p,r)}(X, Y)$. *Then*

$$\begin{aligned} \|T\|_{\mathcal{K}_{(p,r)}} &= \inf \ \|T_y\| \, \|y\| \\ &= \inf \ \|y\| \end{aligned}$$

where the both infimums are taken over all sequences $y = (y_n)_n \in \ell_p(Y)$ *such that*

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n)_n \in B_{\ell_r} \right\}$$

Proof. Let $y = (y_n)_n \in \ell_p(Y)((y_n)_n \in c_0(Y)$ when $p = \infty)$, be such that $T(B_X) \subset \Phi_y(B_{\ell_r})$. We know that $T = \overline{\Phi}_y T_y$, $\|T_y\| \leq 1$, by (Proposition 1.3.3) we have $\left\|\overline{\Phi}_y\right\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \leq \|y\|$

Hence,

$$\begin{aligned} \|T\|_{\mathcal{K}_{(p,r)}} &= \|T\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \\ &\leq \| \,\overline{\Phi}_y\, \|_{\mathcal{N}^{sur}_{(p,1,r^*)}} \| \,T_y\, \| \\ &\leq \| \,T_y\, \| \, \|y\| \\ &\leq \|y\| \,. \end{aligned}$$

Consequently,

$$\|T\|_{\mathcal{K}_{(p,r)}} \leq \inf \|y\| .$$

On the other hand, from the factorization theorem of (t,u,v)-nuclear operators, we know that the $(p,1,r^*)$-nuclear operator TQ_X factorizes as follows:

$$\begin{tikzcd}
Z := \ell_1(B_X) \arrow[r, "Q_X"] \arrow[d, "A"] & X \arrow[r, "T"] & Y \\
\ell_r \arrow[rr, "\Delta"] & & \ell_1 \arrow[u, "B"']
\end{tikzcd}$$

where $\Delta \in \mathcal{L}(\ell_r, \ell_1)$ is a diagonal operator of the form $\Delta(a_n)_n = (a_n\sigma_n)_n$,with $(\sigma_n)_n \in \ell_p((\sigma_n)_n \in c_0$ when $p = \infty)$, $A \in \mathcal{L}(Z, \ell_r)$, $B \in \mathcal{L}(\ell_1, Y)$, and

$$\|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} = \inf \ \|B\| \, \|(\sigma_n)_n\| \, \|A\| ,$$

where the infimum is taken over the all possible factorizations.

Let $\epsilon > 0$. Choose A, $(\sigma_n)_n$, and B as above so that

$$\begin{aligned}
\epsilon + \|T\|_{\mathcal{K}_{(p,r)}} &= \epsilon + \|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} \\
&\geq \|B\| \, \|(\sigma_n)_n\| \, \|A\| = \|(\sigma_n)_n\|
\end{aligned}$$

because we clearly may assume that $\|A\| = \|B\| = 1$. Since $B_X \subset Q_X(B_Z)$ we have

$$T(B_X) \subset (B\Delta A)(B_Z) \subset (B\Delta)(B_{\ell_r}) = \left\{ \sum_{n=1}^{\infty} \alpha_n \sigma_n B(e_n) : (\alpha_n)_n \in B_{\ell_r} \right\},$$

where $(e_n)_n$ is the unit vector basis of $\ell_r(c_0$ when $r = \infty)$. Put

$$y_n = \sigma_n B(e_n).$$

Then $y = (y_n)_n \in \ell_p(Y)((y_n)_n \in c_0$ when $p = \infty)$, $T(B_X) \subset \Phi_y(B_{\ell_r})$, and

$$\|y\| \leq \|(\sigma_n)_n\| .$$

Therefore,

$$\|T\|_{\mathcal{K}_{(p,r)}} \quad \geq \quad \inf \|y\| \ ,$$

this conclues the proof. ■

2.3 Applications to some related operator ideals

Theorem 2.3.1 *Let X and Y be Banach spaces. Assume that $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A linear operator $T : X \to Y$ belongs to $\mathcal{N}^{sur}_{(p,1,r^*)}$ if and only if there exists $y = (y_n)_n \in \ell_p(Y)((y_n)_n \in c_0(Y)$ when $p = \infty)$, such that*

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n)_n \in B_{\ell_r} \right\}.$$

In this case, the $\frac{p}{p+1}$-norm of T is given by

$$\|T\|_{\mathcal{N}^{sur}_{(p,1,r^*)}} = \inf \| \ y \ \|,$$

where the infimum is taken over all admissible sequences $y = (y_n)_n$.

Lemma 2.3.1 *[17, 8.4.4] Let X,Y and Z be Banach spaces, and let $\mathcal{A}$ be an s-Banach operator ideal. If $\|S(x)\| \leq \|T(x)\|$, $x \in X$, for $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{A}(X,Z)$, then $S \in \mathcal{A}^{inj}(X,Y)$ with $\|S\|_{\mathcal{A}^{inj}} \leq \|T\|_{\mathcal{A}}$.*

Theorem 2.3.2 *Let X and Y be Banach spaces. Assume that $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. Let $T : X \to Y$ be a linear operator. Then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{inj}_{(p,r^*,1)}(X,Y)$.

(b) $T \in \mathcal{K}^{dul}_{(p,r)}(X,Y)$.

(c) *There exists $(x^*_n)_n \in \ell_p(X^*)((x^*_n)_n \in c_0(X^*)$ when $p = \infty)$ such that*

$$\| \ T(x) \ \| \leq \| \ (x^*_n(x))_n \ \|_{r^*} \ \forall x \in X. \tag{2.3.1}$$

Moreover, in this case, the $\frac{pr}{p+r}$-norm of T is given by

$$\|T\|_{\mathcal{N}^{inj}_{(p,r^*,1)}} = \|T\|_{\mathcal{K}^{dul}_{(p,r)}} = \inf \| \ (x_n^*)_n \ \|_p$$

where the infimum is taken over all $(x_n^*)_n \in \ell_p(X^*)((x_n^*)_n \in c_0(X^*)$ when $p = \infty)$ satisfying (2.3.1).

Proof. To simplify notation, we omit below the corresponding s-norms.

(a)$\Longrightarrow$(b). We know that $\mathcal{N}_{(p,r^*,1)} \subset \mathcal{N}^{reg}_{(p,r^*,1)}$, the regular hull of $\mathcal{N}_{(p,r^*,1)}$, and by([17, 18.1.6]).

$$\mathcal{N}^{reg}_{(p,r^*,1)} = \mathcal{N}^{dul}_{(p,1,r^*)}$$

Hence, recalling that $\mathcal{A}^{dul\ inj} \subset \mathcal{A}^{sur\ dul}$ for any s-Banach operator ideal $\mathcal{A}$ (see [17, 8.5.9]). By Theorem 2.2.2 . we have

$$\mathcal{N}^{inj}_{(p,r^*,1)} \subset \mathcal{N}^{dul\ inj}_{(p,1,r^*)} \subset \mathcal{N}^{sur\ dul}_{(p,1,r^*)} = \mathcal{N}^{dul}_{(p,1,r^*)}$$

The above inclusion also means that

$$\|T\|_{\mathcal{K}^{dul}_{(p,r)}} \leq \|T\|_{\mathcal{N}^{inj}_{(p,r^*,1)}} \ \forall T \in \mathcal{N}^{inj}_{(p,r^*,1)}$$

(b)$\Longrightarrow$(c). Let $T \in \mathcal{K}^{dul}_{(p,r)}(X, Y)$. Then $T^* \in \mathcal{K}_{(p,r)}(Y^*, X^*)$. Hence (see Theorem 2.2.3), for every $\epsilon > 0$, there exists $(x_n^*)_n \in \ell_p(X^*)$ such that

$$T^*(B_{Y^*}) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n^* : (\alpha_n)_n \in B_{\ell_r} \right\}$$

and

$$\|(x_n^*)_n\|_p \leq \|T\|_{\mathcal{K}^{dul}_{(p,r)}} + \epsilon.$$

For every $x \in X$, we clearly have
$\|T(x)\| = \sup\limits_{\|y^*\| \leq 1} |(T^*y^*)(x)| \leq \sup \left\{ \sum\limits_{n=1}^{\infty} \alpha_n x_n^*(x) : (\alpha_n)_n \in B_{\ell_r} \right\} = \|x_n^*(x)\|_{r^*}$.
It follows that

$$\inf \|(x_n^*)_n\|_p \leq \|T\|_{\mathcal{K}^{dul}_{(p,r)}}$$

whenever the infimum is taken over all $(x_n^*)_n \in \ell_p(X^*)$ satisfying (2.3.1).

(c)$\Longrightarrow$(a). Following an idea from the proof of [17, 8.4.5], we define an operator $A \in \mathcal{L}(X, \ell_{r^*})$ by $A(x) = (x_n^*(x))_n$ $x \in X$. Clearly

$$A(x) = \sum_{n=1}^{+\infty} x_n^*(x) e_n \ , \quad x \in X.$$

Let $(e_n)_n$ be the unit vector basis of $\ell_{r^*} \subset (\ell_r)^*$ (c_0 when $r = 1$), considered as coordinate functionals for ℓ_r. Then we clearly have

$$A = \sum_{n=1}^{+\infty} x_n^* \otimes e_n$$

and from

$$x_n^* \otimes e_n = \|(x_n^*)_n\| \frac{x_n^*}{\|(x_n^*)_n\|} \otimes e_n$$

we have that and $(\|(x_n^*)_n\|)_n \in \ell_p$. (respectively $(\|(x_n^*)_n\|)_n \in c_0$ when $p = \infty$), $(\|x_n^*\|^{-1} x_n^*)_n \in \ell_\infty(X^*) = \ell_\infty^w(X^*)$, $(e_n)_n \in \ell_r^w(\ell_{r^*}) \subset \ell_r^w(\ell_r^*)$ (respectively, $(e_n)_n \in \ell_1^w(c_0) \subset \ell_1^w(\ell_\infty)$ when $r = 1$) (cf. Proposition 1.3.2). Hence (see Definition 1.3.1) $A \in \mathcal{N}_{(p,r^*,1)}(X, \ell_{r^*})$, and

$$\|A\|_{\mathcal{N}_{(p,r^*,1)}} \leq \|(x_n^*)_n\|_p \, .$$

Since $\|T(x)\| = \|A(x)\|$ for all $x \in X$, by Lemma 2.3.1, we immediately get that $T \in \mathcal{N}^{inj}_{(p,r^*,1)}(X, Y)$ and

$$\|T\|_{\mathcal{N}^{inj}_{(p,r^*,1)}} \leq \|A\|_{\mathcal{N}_{(p,r^*,1)}} \leq \|(x_n^*)_n\|_p \, .$$

This proves the theorem. ■

Corollary 2.3.1 *Let $1 \leq p \leq 2$. Then*

$$\mathcal{K}_{(p,2)} = \mathcal{N}_{(p,1,2)},$$

and

$$\mathcal{K}^{dul}_{(p,2)} = \mathcal{N}_{(p,2,1)}$$

as $\frac{2p}{2+p}$-Banach operator ideals.

Chapter 3

Lipschitz (p, r)-compact mappings

In this chapter, we extend the notion of some well-known forms of Lipschitz p-compact operators which has been studied in [1] to Lipschitz (p, r)-compact operators. Also we transfer some properties of the linear case into the Lipschitz case. Finally we define the notions of Lipschitz-free (p, r)-compact operators and Lipschitz locally (p, r)-compact operators.

3.1 Notation and preliminaries

Let X and Y will be pointed metric spaces with a base point denoted by 0 and metric will be denoted by d. Let E and F Banach spaces over the same field $\mathbb{K}$.

We recall that a Banach space E will be considered as pointed metric spaces with distinguished point 0 and distance $d(x,x') = \|x - x'\|$. The Lipschitz space $\mathrm{Lip}_0(X,E)$ is the Banach space of all Lipschitz mappings T from X to E that vanish at 0, under the Lipschitz norm

$$\mathrm{Lip}\,(T) = \inf\{C > 0\colon\ \|T(x) - T(x')\| \leq Cd(x,x');\ \forall x, x' \in X\}.$$

When $E = \mathbb{K}$, $\mathrm{Lip}_0(X,\mathbb{K})$ is denoted by $X^{\#}$ and it is called the Lipschitz dual of X.

It is clear that $\mathcal{L}(E,F)$ is a subspace of $\mathrm{Lip}_0(E,F)$ and, in particular, E^* is a subspace of $E^{\#}$. One of the main tools that we will use is the Lipschitz-free Banach space of a metric space X, $\mathcal{F}(X)$ (also known as the Arens–Ells space). For $x \in X$, denote by δ_x the function $\delta_x : X^{\#} \longrightarrow \mathbb{K}$ defined as

$$\delta_x(f) = f(x),\ \ f \in X^{\#}.$$

Then $\mathcal{F}(X)$ is the closed linear span of $\{\delta_x, x \in X\}$ in $\left(X^{\#}\right)^*$. In [10] or [11], it is proved that $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$.

We summarize some basic properties concerning Lipschitz-free Banach spaces in the following lemma. This can be found for instance in [15]. Will use it without further mentioning.

Lemma 3.1.1 *Let X, Y be pointed metric spaces and E be a Banach space.*

1. The dual space of $\mathcal{F}(X)$ is isometrically isomorphic to $X^{\#}$ through the mapping $Q_X : X^{\#} \to \mathcal{F}(X)^*$ given by

$$Q_X(f)(\gamma) = \gamma(f), f \in X^{\#}, \gamma \in \mathcal{F}(X).$$

2. For any Lipschitz mapping $T \in \mathrm{Lip}_0(X, Y)$ there exists a unique linear map $\widehat{T}$: $\mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\widehat{T}\delta_X = \delta_Y T$. That is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \delta_x \downarrow & & \downarrow \delta_y \\ \mathcal{F}(X) & \xrightarrow{\widehat{T}} & \mathcal{F}(Y) \end{array}$$

3. There exists a quotient map $\beta_E \colon \mathcal{F}(E) \to E$ such that $\beta_E \circ \delta_E = Id_E$.

4. For any Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ there exists a unique linear map T_L : $\mathcal{F}(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$. That is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_x \downarrow & \nearrow T_L & \\ \mathcal{F}(X). & & \end{array}$$

Moreover $\|T_L\| = \mathrm{Lip}(T)$.

in particular, from (2) and (3) we can deduce (4) with $T_L = \beta_E \widehat{T}$.

The map $\beta_E \colon \mathcal{F}(E) \to E$ is called the barycenter map. For $T \in \mathrm{Lip}_0(X, E)$, we will consider the Lipschitz adjoint (or dual) of T, defined by Sawashima [20]. That is, $T^{\#} \in \mathcal{L}(E^{\#}, X^{\#})$ is the linear operator given by

$$T^{\#}(g) = g \circ T$$

for all $g \in E^{\#}$. The restriction of $T^{\#}$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t. Is clear that $Q_X \circ T^t = T_L^*$, where T_L^* denotes the transpose map of the linearization of T and also that $\widehat{T}^* \circ Q_Y = Q_X \circ T^{\#}$.

We refer to the reader to Weaver's book [22] for the basics of Lipschitz operators.

3.2 The composition ideal of Lipschitz (p, r)-compact operators

Recall that, for $T \in \mathrm{Lip}_0(X, E)$, Jiménez–Vargas, Sepulcre and Villegas–Vallecillos defined in [13] the Lipschitz image of T as

$$Im_{\mathrm{Lip}}(T) = \left\{ \frac{T(x) - T(y)}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. In [16] Lassalle and all third author defined the measure of the size of the p-compact set K as

$$m_p(K, E) = \inf \left\{ \|(x_n)_n\|_p : K \subset p\text{-conv}\,\{(x_n)_n\} \right\}.$$

and $m_p(K, E) = \infty$ if K is not p-compact.

If one replaces the p-convex hull of a p-summable sequence by the (p, r)-convex hull then we define the measure of the size of the (p, r)-compact set K by

$$m_{(p,r)}(K, E) = \inf \left\{ \|(x_n)_n\|_p : K \subset (p, r)\text{-conv}\,\{(x_n)_n\} \right\}.$$

and $m_{(p,r)}(K, E) = \infty$ if K is not (p, r)-compact. When the context $K \subset E$ is understood, we simply write $m_{(p,r)}(K)$ instead of $m_{(p,r)}(K, E)$, and we have the following equalities:

$$m_{(p,r)}(K) = m_{(p,r)}(\overline{K}) = m_{(p,r)}(\Gamma(K)), \tag{3.2.1}$$

where $\Gamma(K)$ denotes the absolutely convex hull of K, a relatively (p, r)-compact set.

If $r = p^*$ the measure of the size of the (p, p^*)-compact is precisely the measure of the size of the p-compact in other words

$$m_{(p,p^*)}(K, E) = m_p(K, E).$$

Because the p-conv$\{(x_n)_n\}$ is precisely the (p, p^*)-conv$\{(x_n)_n\}$, and $m_{(\infty,1)}(K, E) = \sup\limits_{x \in K} \|x\|$.

The space of all (p, r)-compact linear operators from E to F $\mathcal{K}_{(p,r)}(E, F)$ becomes a Banach space if we endow it with the norm

$$\|T\|_{\mathcal{K}_{(p,r)}} = \mathcal{K}_{(p,r)}(T) = m_{(p,r)}(T(B_E)).$$

In [1] Achour, Dahia, and Turco defined the Lipschitz p-compact operators in the following way

Let X be a pointed metric space, E a Banach space and let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ is Lipschitz p-compact if its Lipschitz image is relatively p-compact. We denote by $\mathcal{K}_p^L(X, E)$ the set of all Lipschitz p-compact mappings from X to E. Moreover, if $T \in \mathcal{K}_p^L(X, E)$, then

$$k_p^L(T) = m_p(Im_{\mathrm{Lip}}(T)).$$

If $p=\infty$, then $\mathcal{K}^L_\infty(X,E)=\mathrm{Lip}_{0K}(X,E)$, the space of Lipschitz compact operators defined and studied in [13].

We define Lipschitz (p,r)-compact in an obvious way

Definition 3.2.1 *Let $1\leq p\leq\infty$, $1\leq r\leq p^*$. A Lipschitz operator $T\in\mathrm{Lip}_0(X,E)$ is Lipschitz (p,r)-compact if its Lipschitz image is relatively (p,r)-compact.*

We denote by $\mathcal{K}^L_{(p,r)}(X,E)$ the set of all Lipschitz (p,r)-compact mappings from X to E. Moreover, if $T\in\mathcal{K}^L_{(p,r)}(X,E)$, then we set

$$k^L_{(p,r)}(T)=m_{(p,r)}(Im_{\mathrm{Lip}}(T)).$$

For the extremal cases, it is clear that $\mathcal{K}^L_{(\infty,1)}(X,E)=\mathcal{K}^L_\infty(X,E)=\mathrm{Lip}_{0K}(X,E)$, the space of Lipschitz compact operators defined and studied in [13]. And $\mathcal{K}^L_{(p,p^*)}(X,E)=\mathcal{K}^L_p(X,E)$ the space of Lipschitz compact operators defined and studied in [1]

Remark 3.2.1 *The Lipschitz (p,r)-compact operators can be seen as an extension of the linear (p,r)-compact operators. Indeed, for E and F Banach spaces, for any linear operator $T\colon E\to F$ we have that the absolutely convex hull of $Im_{\mathrm{Lip}}(T)$ coincides with $T(B_E)$. Then, since a set is (p,r)-compact if and only if its absolutely convex hull is also (p,r)-compact with the same measure.*

We extend the results obtained in [1] of $\mathcal{K}^L_p$ the class of Lipschitz p-compact operator to the more general case of $\mathcal{K}^L_{(p,r)}$ the class of Lipschitz (p,r)-compact operator.

Proposition 3.2.1 *Let E and F be Banach spaces, and $1\leq p\leq\infty$, $1\leq r\leq p^*$. Then a linear operator $T\in\mathcal{L}(E,F)$ is (p,r)-compact if and only if it is Lipschitz (p,r)-compact. Moreover, we have*

$$k^L_{(p,r)}(T)=k_{(p,r)}(T).$$

From Theorem 2.1.2 we immediately get the following inclusion results.

Proposition 3.2.2 *Let X be a pointed metric and E be a Banach space. Let $1\leq p\leq q\leq\infty$, $1\leq r\leq p^*$, and $1\leq s\leq q^*$. Assume that*

$$\frac{1}{q}+\frac{1}{s}\leq\frac{1}{p}+\frac{1}{r}.$$

Then the Lipschitz (p, r)-compact operators are Lipschitz (q, s)-compact operators, and

$$k^L_{(q,s)}(T) \leq k^L_{(p,r)}(T),$$

for any $T \in \mathcal{K}^L_{(p,r)}(X, E)$.

In particular, the Lipschitz $(p, 1)$-compact operators are Lipschitz $(q, 1)$ compact.

Theorem 3.2.1 *Let X be a pointed metric space, E a Banach space and let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. An operator $T \in \mathrm{Lip}_0(X, E)$ is Lipschitz (p, r)-compact if and only if its linearization $T_L \colon \mathcal{F}(X) \to E$ is linear (p, r)-compact. Moreover, we have*

$$k^L_{(p,r)}(T) = k_{(p,r)}(T_L). \tag{3.2.2}$$

Proof. In the proof of [13, Proposition 2.1] it is shown that, for $T \in \mathrm{Lip}_0(X, E)$,

$$Im_{\mathrm{Lip}}(T) \subset T_L(B_{\mathcal{F}(X)}) \subset \overline{\Gamma}\left(Im_{\mathrm{Lip}}(T)\right). \tag{3.2.3}$$

Then, by (3.2.1) we get that

$$m_{(p,r)}(Im_{\mathrm{Lip}}(T)) = m_{(p,r)}(T_L(B_{\mathcal{F}(X)})) = m_{(p,r)}(\overline{\Gamma}\left(Im_{\mathrm{Lip}}(T)\right)),$$

and the result follows. ■

Let $0 < s \leq 1$. The notion of (s-Banach) Lipschitz operator ideal was introduced by Cabrera-Padilla ,Chàvez-Domínguez, Jiménez-Vargas and Villegas-Vallecillos [8]. This can be seen as an extension of the linear (s-Banach) operator ideal.

Definition 3.2.2 *A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components*

$$\mathcal{I}_{Lip}(X, E) := \mathrm{Lip}_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy

(i) $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $\mathrm{Lip}_0(X, E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X, E)$ for $v \in E$ and $g \in X^{\#}$.

(iii) The ideal property: if $S \in \mathrm{Lip}_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y, F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a s-normed (s-Banach) Lipschitz operator ideal if there is $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X, E), \|.\|_{\mathcal{I}_{Lip}})$ is a s-normed (s-Banach) space and $\mathrm{Lip}(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in \mathrm{Lip}_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then

$$\|wTS\|_{\mathcal{I}_{Lip}} \leq \mathrm{Lip}(S) \|T\|_{\mathcal{I}_{Lip}} \|w\| .$$

Clearly, the generic Lipschitz operator 1-Banach ideal is precisely the Lipschitz operator Banach ideal introduced in [3].

Following [3, Definition 3.1], there is a way to construct a (s-Banach) Lipschitz operator ideal from a (s-Banach) linear operator ideal, called *composition method*. Let $\mathcal{A}$ be a (s-Banach) linear operator ideal. A Lipschitz mapping $T \in \mathrm{Lip}_0(X, E)$ belongs to the *composition Lipschitz ideal* $\mathcal{A} \circ \mathrm{Lip}_0$ if there exists a Banach space F, a *Lipschitz operator* $S \in \mathrm{Lip}_0(X, F)$ and a linear operator $u \in \mathcal{A}(F, E)$ such that $T = uS$. If $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is a s-Banach operator ideal we write

$$\|T\|_{\mathcal{A} \circ \mathrm{Lip}_0} = \inf \|u\|_{\mathcal{A}} \mathrm{Lip}(S),$$

where the infimum is taken over all u and S as above.

In [3], the authors establish a criterion to decide whenever a Lipschitz operator ideal is of composition or not.

Proposition 3.2.3 *[3, Proposition 3.2] Let X be a pointed metric space, E a Banach space and $\mathcal{A}$ an (s-Banach) operator ideal. A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ belongs to $\mathcal{A} \circ \mathrm{Lip}_0(X, E)$ if and only if its linearization T_L belongs to $\mathcal{A}(\mathcal{F}(X), E)$.*

Furthermore, if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a s-Banach operator ideal then $(\mathcal{A} \circ \mathrm{Lip}_0, \|\cdot\|_{\mathcal{A}\circ\mathrm{Lip}_0})$ is s-Banach Lipschitz operator ideal with

$$\|T\|_{\mathcal{A}\circ\mathrm{Lip}_0} = \|T_L\|_{\mathcal{A}} .$$

By Theorem 3.2.1 and the above criterion, we have the following.

Proposition 3.2.4 *Let $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$. The class $\mathcal{K}^L_{(p,r)}$ is the s-Banach Lipschitz operator ideal generated by the composition method from the Banach operator ideal $\mathcal{K}_{(p,r)}$. In other words*

$$\mathcal{K}^L_{(p,r)}(X, E) = \mathcal{K}_{(p,r)} \circ \mathrm{Lip}_0(X, E) \quad \textit{isometrically},$$

for every X pointed metric space and E a Banach space.

Let $\mathcal{I}_{Lip}$ be a Lipschitz operator ideal. We recall that a Lipschitz operator $T \in \mathrm{Lip}_0(X, Y)$ belongs to the injective hull of $\mathcal{I}_{Lip}$ if there exists a pointed metric space Z, and a Lispchitz operator $S \in \mathrm{Lip}_0(X, Z)$ such that

$$d(T(x), T(x_0)) \leq d(S(x), S(x_0))$$

for all $x, x_0 \in X$.

We denote by $\mathcal{I}^{inj}_{Lip}(X, Y)$. The class of all operators from X to Y which belongs to the injective hull of $\mathcal{I}_{Lip}$ (see [2]).

Proposition 3.2.5 *[2, Proposition 2.1]Let $\mathcal{A}$ be a (s-Banach) linear operator ideal. Then*

$$(\mathcal{A}\circ\mathrm{Lip}_0)^{inj} = \mathcal{A}^{inj}\circ\mathrm{Lip}_0.$$

In particular, if $\mathcal{I}_{\mathrm{Lip}}$ is a (s-Banach) Lipschitz operator ideal of composition type, then it is also $\mathcal{I}^{inj}_{\mathrm{Lip}}$.

Proof. Fix X and E and take $T \in (\mathcal{A}\circ\mathrm{Lip}_0)^{inj}(X, E)$. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "\delta_X"'] & E \arrow[r, "\iota_E"] & \ell_\infty(B_{E^*}) \\
\text{Æ}(X) \arrow[ur, "T_L"] \arrow[urr, "(\iota_E T)_L"'] & &
\end{tikzcd}$$

Note that, since $\iota_E : E \to \ell_\infty(B_{E^*})$ is a linear operator, the uniqueness of the linearization maps gives that $(\iota_E T)_L = \iota_E T_L \in \mathcal{A}(Æ(X), \ell_\infty(B_{E^*}))$. Then, we have that $T \in (\mathcal{A}\circ\mathrm{Lip}_0)^{inj}(X,E)$ if and only if $\iota_E T \in (\mathcal{A}\circ\mathrm{Lip}_0)(X, \ell_\infty(B_{E^*}))$ if and only if $(\iota_E T)_L \in \mathcal{A}(Æ(X), \ell_\infty(B_{E^*}))$. This is equivalent to the fact that operator $\iota_E T_L$ belongs to the ideal $\mathcal{A}$, or, in other words, to $T_L \in \mathcal{A}^{inj}(Æ(X), E)$. But this last is, by definition the same to $T \in \mathcal{A}^{inj}\circ\mathrm{Lip}_0(X,E)$. The isometry follows in the same way. ■

Let $0 < s \leq 1$. The Lipschitz dual of an (s-Banach)operator ideal $\mathcal{A}$ is defined as follows.

$$\mathcal{A}^{\mathrm{Lip}-dul}(X,E) = \left\{T \in \mathrm{Lip}_0(X,E) : T^t \in \mathcal{A}(E^*, X^{\#})\right\},$$

for a pointed metric space X and a Banach space E.

If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a s-normed (s-Banach) operator ideal, we define

$$\|T\|_{\mathcal{A}^{\mathrm{Lip}-dul}} = \left\|T^t\right\|_{\mathcal{A}}.$$

Then $(\mathcal{A}^{\mathrm{Lip}-dul}, \|.\|_{\mathcal{A}^{\mathrm{Lip}-dul}})$ becomes a s-normed (s-Banach) Lipschitz ideal (see [3, Definition 3.8]).

To introduce the Lipschitz dual of (p,r)-compact operators we introduce the most important results obtained in [5] for the strongly Lipschitz (p,r,s)-nuclear operators.

Let $(f_j)_j$ be a sequence in $X^{\#}$. We say that $(f_j)_j$ is Lipschitz-ω^*-p-summable if there is a constant C such that for all $n \in \mathbb{N}$ and for all $x, x' \in X$ we have

$$\left\|(f_j(x) - f_j(x'))_{j=1}^n\right\| \leq cd(x,x').$$

The smallest such constant C will be denoted by $\omega_{s^*}^{L,\omega^*}((f_j)_j)$, and $l_{s^*}^{L,\omega^*}(X^{\#})$ the set of all Lipschitz-ω^*-p-summable sequences in $X^{\#}$.

Definition 3.2.3 *Suppose that,* $\frac{1}{s} + \frac{1}{r} \leq 1 + \frac{1}{p}$ *and* $T \in \mathrm{Lip}_0(X,E)$. *We say that*

$T : X \longrightarrow E$ is called strongly Lipschitz (p,r,s)-nuclear if T can be written in the form

$$T = \sum_j \lambda_j f_j \otimes v_j \tag{3.2.4}$$

with $(\lambda_j)_j \in \ell_p((\lambda_j)_j \in c_0$ when $p = \infty)$, $(f_j)_j \in l_{s^*}^{L,\omega^*}(X^{\#})$, and $(v_j)_j \in l_{r^*}^{\omega}(E)$.

The set of all strongly Lipschitz (p,r,s)-nuclear operators is denoted by $\mathcal{SN}_{(p,r,s)}^L$.

We put

$$sN^L_{(p,r,s)}(T) := \inf \|(\lambda_j)_j\|_p \, \omega^{L,\omega^*}_{s_*}((f_j)_j) \, \|(v_j)_j\|^w_{u^*}$$

where the infimum is taken over all strongly Lipschitz (p, r, s)-nuclear representations (3.2.4).

Theorem 3.2.2 *[5] Let $\frac{1}{\beta} := \frac{1}{p} + \frac{1}{s^*} + \frac{1}{r^*} \geq 1$. Then $(\mathcal{SN}^L_{(p,r,s)}, sN^L_{(p,r,s)}(.))$ is the β-Banach Lipschitz operator ideal, generated by the composition method from the Banach operator ideal $\mathcal{N}_{(p,r,s)}$. In other words*

$$\mathcal{SN}^L_{(p,r,s)}(X, E) = \mathcal{N}_{(p,r,s)} \circ \mathrm{Lip}_0(X, E) \quad \textit{isometrically.}$$

For every X pointed metric space and E a separable Banach space.

Proposition 3.2.6 *Let X be a metric space, and E separable Banach space. Assume that $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. Let $T \in \mathrm{Lip}_0(X, E)$. Then the following statements are equivalent:*

(a) $T \in (\mathcal{SN}^L_{(p,r^*,1)})^{inj}(X, E)$.

(b) $T \in \mathcal{K}^{L-dul}_{(p,r)}(X, E)$.

Proof. Let $T \in (\mathcal{SN}^L_{(p,r^*,1)})^{inj}(X, E)$. By Proposition 3.2.5 we have

$$(\mathcal{SN}^L_{(p,r^*,1)})^{inj}(X, E) = (\mathcal{N}_{(p,r^*,1)})^{inj} \circ \mathrm{Lip}_0(X, E)$$

Then , $T \in (\mathcal{N}_{(p,r^*,1)})^{inj} \circ \mathrm{Lip}_0(X, E)$ if and only if its linearization $T_L \in (\mathcal{N}_{(p,r^*,1)})^{inj}(\mathcal{F}(X), E)$(see Proposition 3.2.3). By Theorem 2.3.2 this is equivalent to that $T_L \in \mathcal{K}^{dul}_{(p,r)}(\mathcal{F}(X), E)$, then $T_L^* \in \mathcal{K}_{(p,r)}(E^*, \mathcal{F}(X)^*)$. On the other hands, we have the equality $T_L^* = Q_X \circ T^t$, where $Q_X : X^\# \to \mathcal{F}(X)^*$ is the canonical isometric isomorphism, and $T^t : E^* \longrightarrow X^\#$ the Lipschitz transpose map of T. Then

$$T^t = Q_X^{-1} \circ T_L^* \in \mathcal{K}_{(p,r)}(E^*, X^\#).$$

The result follows. ∎

3.3 Lipschitz-free and locally (p, r)-compact mappings

This section is devoted to another two classes of Lipschitz mappings related with (p, r)-compact sets. The first class we discuss is motivated by notion of Lipschitz-free p-compact operators, recently introduced by Cabrera–Padilla and Jiménez–Vargas in [7]. And Note that, in particular, this class can be defined for Lipschitz operators between metric spaces.

Definition 3.3.1 *Let X and Y be pointed metric spaces and $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A Lipschitz operator $T \in \mathrm{Lip}_0(X, Y)$ is called Lipschitz-free (p, r)-compact if the mapping $\delta_Y \circ T : X \to \mathcal{F}(Y)$ is a Lipschitz (p, r)-compact operator. The set of all Lipschitz-free (p, r)-compact operators between X and Y will be denoted by $\mathcal{FK}^L_{(p,r)}(X, Y)$.*

For the extremal cases, the Lipschitz-free $(\infty, 1)$-compact mappings coincide with the Lipschitz-free compact operators of Cabrera–Padilla and Jiménez–Vargas, and the Lipschitz-free (p, p^*)-compact mappings coincide with the Lipschitz-free p-compact defined in [1].

Theorem 3.3.1 *Let X and Y be pointed metric spaces and $T \in \mathrm{Lip}_0(X, Y)$. For $1 \leq p \leq \infty$, $1 \leq r \leq p^*$, the following are equivalent*

1. T is Lipschitz-free (p, r)-compact.
2. The operator $\widehat{T} \colon \mathcal{F}(X) \to \mathcal{F}(Y)$ is (p, r)-compact.

Proof. Applying Theorem 3.2.1, we get that T is Lipschitz-free (p, r)-compact if and only if $(\delta_Y \circ T)_L \colon \mathcal{F}(X) \to \mathcal{F}(Y)$ is a (p, r)-compact operator. Then, the equivalence between (1) and (2) follows by noticing that the operator $(\delta_Y \circ T)_L$ coincides with the operator $\widehat{T}$. ■

The other class that we will deal in this section is the class of Lipschitz locally (p, r)-compact operators, whose definition arise in a natural way.

Definition 3.3.2 *Let X be a pointed metric space, E a Banach space and let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ is called Lipschitz locally (p, r)-compact at $x_0 \in X$ if there exists $\varepsilon > 0$ such that $T(B_\varepsilon(x_0))$ is a relatively (p, r)-compact set in E. The operator T is said to be Lipschitz locally (p, r)-compact if it is locally (p, r)-compact at x_0 for every $x_0 \in X$. We denote by $\mathcal{K}^{Loc}_{(p,r)}(X, E)$ the set of all locally Lipschitz (p, r)-compact operators.*

Remark 3.3.1 *We have that a linear operator between Banach spaces is (p, r)-compact if and only if maps bounded sets into relatively (p, r)-compact sets. Then, every linear operator is (p, r)-compact if and only if is Lipschitz locally (p, r)-compact.*

Remark 3.3.2 *Since every relatively (p, r)-compact set is (q, r)-compact whenever $1 \leq p \leq q \leq \infty$, $1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. We assume that*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r},$$

then we have that $\mathcal{FK}^L_{(p,r)} \subset \mathcal{FK}^L_{(q,s)}$ and $\mathcal{K}^{Loc}_{(p,r)} \subset \mathcal{K}^{Loc}_{(q,s)}$.

In particular, $\mathcal{FK}^L_{(p,1)} \subset \mathcal{FK}^L_{(q,1)}$ and $\mathcal{K}^{Loc}_{(p,1)} \subset \mathcal{K}^{Loc}_{(q,1)}$.

The next result describes the relationship between the three classes of Lipschitz operators that we introduced.

Proposition 3.3.1 *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. Then*

$$\mathcal{FK}^L_{(p,r)} \subset \mathcal{K}^L_{(p,r)} \subset \mathcal{K}^{Loc}_{(p,r)}$$

Proof. Fix a pointed metric space X and a Banach space E and take $T \in \mathcal{FK}^L_{(p,r)}(X,E)$. Then $\delta_E \circ T$ is a Lipschitz (p,r)-compact operator. That is, the Lipschitz image of $\delta_E \circ T$ is a relatively (p,r)-compact set in $\mathcal{F}(E)$. As in the proof of [7, Proposition 2.2], we have the equality $Im_{\mathrm{Lip}}(T) = \beta_E(Im_{\mathrm{Lip}}(\delta_E \circ T))$, where β_E is the barycenter map. Thus $Im_{\mathrm{Lip}}(T)$ is a relatively (p,r)-compact set in E and, in particular, T is a Lipschitz (p,r)-compact operator. Then, we showed that $\mathcal{FK}^L_{(p,r)} \subset \mathcal{K}^L_{(p,r)}$.

For the other inclusion, if $S \in \mathcal{K}^L_{(p,r)}(X,E)$, then its linearization is a (p,r)-compact operator. Then, for any $x \in X$ and any $\epsilon > 0$, we have $T(B_\epsilon(x)) = T_L(\delta_X(B_\epsilon(x)))$. Since $\delta_X(B_\epsilon(x))$ is a bounded set of $\mathcal{F}(X)$, we conclude that $T(B_\epsilon(x))$ is a (p,r)-compact set. Thus $\mathcal{K}^L_{(p,r)} \subset \mathcal{K}^{Loc}_{(p,r)}$. ∎

Theorem 3.3.2 *(The strong ideal property) Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. For all pointed metric spaces X, Y, Z, W, if $R \in \mathrm{Lip}_0(X,Y)$, $T \in \mathcal{FK}^L_{(p,r)}(Y,Z)$ and $S \in \mathrm{Lip}_0(Z,W)$ then $S \circ T \circ R \in \mathcal{FK}^L_{(p,r)}(X,W)$.*

Proof. We have to prove that $\widehat{S \circ T \circ R} \colon \mathcal{F}(X) \to \mathcal{F}(W)$ is a (p,r)-compact operator. It is straightforward to check that $\widehat{S \circ T \circ R} = \widehat{S} \circ \widehat{T} \circ \widehat{R}$. The result follows since $\widehat{T} \colon \mathcal{F}(Y) \to \mathcal{F}(Z)$ is a (p,r)-compact linear operator. ∎

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الملخص

في هذا العمل سنهتم بدراسة مفهوم مجموعات متراصة نسبيا , و دراسة فضاء المؤثرات (p,r)-متراصة و وصف هيكله باعتباره مثالى, كما نقدم مفهوم مؤثر ليبشيتز (p,r)-متراص و نبين أنه يمكن إعتباره إمتدادا طبيعيا للحالة الخطية ونحول بعض الخصائص الحالة الخطية إلى الحالة اليبشيتزية

الكلمات المفتاحية. مؤثر متراص , مجموعة متراصة نسبيا , مثالي المؤثر, مؤثر (p,r)-متراص , مؤثر ليبشيتز p-متراص

Résumé

Dans ce travail on s'intéresse par la notion des ensembles relativement compacts. Nous allons ètudier la classe des operateurs (p,r)-compact et dècrire sa structure comme un idèal. Nous introduisons aussi le concept des operateurs Lipschitz (p,r)-compact, on montrera que cela peut être vu comme une extension naturelle du cas linéaire et on transférera certaines propriétés du cas linéaire au cas Lipschitzien.

Mots-clés. Ensemble relativement compact , Opérateurs compacts, Operateurs (p,r)-compact, Opérateurs Idéal, Operateurs Lipschitzien p-compact.

Abstract

In this work we will interest by the notion of relatively compact sets, we go on to study the space of (p,r)-compact operators and describe its structure as an operator ideal. We introduce the concept of Lipschitz (p,r)-compact operators. We show that can be seen as a natural extension of the linear case and we transfer some properties of the linear case into the Lipschitz setting.

Key-words. Compact operators, (p,r)-compact operators, Ideal operators, Lipschitz p-compact operators, Relatively (p,r)-compact subsets.